Ultracoproducts of G-flows

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If X and Y are G-spaces, a map $\varphi\colon X \to Y$ is a G-map if it is continuous and G-equivariant.

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Definition

If X is a G-space and $f \in \mathrm{C}_b(X)$, call f G-continuous if the map $\lambda_f : G \to \mathrm{C}_b(X)$ given by $\lambda_f(g) = f \cdot g$ is norm continuous.

Write $\mathrm{C}_G(X)$ for the algebra of G-continuous functions on X.

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Write $\mathrm{C}_G(X)$ for the algebra of G-continuous functions on X.

Fact: If X is a G-flow, then $\mathrm{C}_G(X) = \mathrm{C}_b(X)$.

Definition

If X is a G-space, the maximal G-equivariant compactification is a G-flow $\alpha_G(X)$ and a G-map $\iota_{G,X}\colon X \to \alpha_G(X)$ such that if Y is any G-flow and $\varphi\colon X \to Y$ is a G-map, then there is $\widetilde{\varphi}\colon \alpha_G(X) \to Y$ with $\varphi = \widetilde{\varphi} \circ \iota_{G,X}$.

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While the map $\iota_{G,X}$ need not be injective, in this talk we will almost always be in situations where $\iota_{G,X}$ is an embedding, in which case we simply identify $X \subseteq \alpha_G(X)$.

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Definition

Let $\mathcal{U} \in \beta I \setminus I$. We define the ultracoproduct of $\{X_i : i \in I\}$ along $\mathcal{U}$ to be the G-flow $\Sigma^G_{\mathcal{U}} X_i := \pi_I^{-1}(\{\mathcal{U}\})$. If $X_i \cong X$ for every $i \in I$, we call $\Sigma^G_{\mathcal{U}} X$ the ultracopower of X along $\mathcal{U}$.

Equivalently, take $\mathrm{C}_G(\bigsqcup_{i\in I} X_i)$ and form $\sim_{\mathcal{U}}$, where given $e = (e_i)_{i\in I}, f = (f_i)_{i\in I} \in \mathrm{C}_G(X)$, we set

$$e \sim_{\mathcal{U}} f \Leftrightarrow \forall\, \epsilon > 0\, \{i \in I : |e_i - f_i| < \epsilon\} \in \mathcal{U}.$$

Then $\Sigma^G_{\mathcal{U}} X_i$ is the Gelfand dual of $\mathrm{C}_G(\bigsqcup_{i\in I} X_i)/\sim_{\mathcal{U}}$.

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In modern language, Bankston ultracoproducts are simply the dual of ultraproducts of commutative C*-algebras viewed as continuous structures.

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By a similar argument to theirs, these coincide when G is locally compact, but in general, these can be different.

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If X is compact, write $\mathrm{V}(X)$ for the Vietoris hyperspace of compact subspaces of X. A typical basic open neighborhood has the form

$$\{Y \in \mathrm{V}(X) : Y \subseteq A_0 \cup \cdots \cup A_{n-1} \text{ and } \forall i < n\,(Y \cap A_i \neq \emptyset)\}$$

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If X is a G-flow, write
$\mathrm{Sub}_G(X) = \{Y \in \mathrm{V}(X) : Y \text{ a } G\text{-subflow}\}$.

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- $\mathrm{TT}_G(X) := \{Y \in \mathrm{Sub}_G(X) : Y \text{ is top. trans.}\}$.

  Recall that a G-flow is topologically transitive if every open G-invariant subspace is dense.

Motivating question: How complicated are $\mathrm{Min}_G(X)$ and $\mathrm{TT}_G(X)$ as subspaces of $\mathrm{Sub}_G(X)$?

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Fact: If $\{X_i : i \in I\}$ are G-flows and $\mathcal{U} \in \beta I \setminus I$, then $\lim_{i \to \mathcal{U}} X_i = \Sigma^G_{\mathcal{U}} X_i$ in $\mathrm{Sub}_G(\alpha_G(\bigsqcup_{i \in I} X_i))$.

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If X is a G-flow and $(X_i)_{i\in I}$ is a net of subflows of X with $\lim_i X_i = Y$, then if $\mathcal{U} \in \beta I \setminus I$ is any cofinal ultrafilter, then $\Sigma_{\mathcal{U}}^G X_i$ factors onto Y.

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So to show that certain subspaces of $\mathrm{Sub}_G(X)$ are well-behaved, it often suffices to show that the defining property is well-behaved under ultraproducts.

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Conversely, to show that certain subspaces are not well-behaved, ultracoproducts can often serve as counterexamples.

Definition (Basso-Z. 2021)

A topological group G is called CAP if for every G-flow X, $\mathrm{Min}_G(X) \subseteq \mathrm{Sub}_G(X)$ is closed.

Equivalently, this occurs iff $\bigcup \mathrm{Min}_G(X) := \mathrm{AP}_G(X) \subseteq X$ is closed. Thus CAP stands for "closed almost-periodic."

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Ultracoproducts simplify both directions of the argument.

Theorem (Z.)

If G is a Polish group with $\mathrm{M}(G)$ metrizable, then every ultracopower of $\mathrm{M}(G)$ is isomorphic to $\mathrm{M}(G)$.

Conversely, if G is Polish and $\mathrm{M}(G)$ is not metrizable, then there is $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ such that the corresponding ultracopower is non-metrizable.

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While (1) $\Rightarrow$ (2) is easiest to prove directly, for (2) $\Rightarrow$ (1), one constructs an ultracopower of the Samuel compactification of G which is not topologically transitive.

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Theorem (Basso-Z. 2023+)

Given a Polish group G, the following are equivalent.

- *G is GPP.*
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- *For any G-flow X and any $Y \in \overline{\mathrm{Min}_G(X)}$, Y contains a unique minimal subflow.*

For $G = \mathrm{Aut}(\mathbb{Q})$ and $X = 2^{\mathbb{Q}}$, the collection of subflows of X containing a unique minimal subflow is not Vietoris closed.

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Despite the suggestive notation, it is far from obvious that $\preceq_G$ is transitive or that $\sim_G$ is an equivalence relation.

The problem: Given sets I and J, $\mathcal{U} \in \beta I$, $\mathcal{V} \in \beta J$, and a G-flow Z, need $\Sigma_{\mathcal{U}}^G \Sigma_{\mathcal{V}}^G Z \cong \Sigma_{\mathcal{U}\otimes\mathcal{V}}^G Z$. The former factors onto the latter.

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Fact: If X is a G-space, then $f \in \mathrm{C}_G(X)$ iff there is some continuous right-invariant pseudometric d on G such that f is d-orbit-Lipschitz, i.e. $\forall x \in X\ (|f(gx) - f(x)| \leq d(g, 1_G))$.

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Question (G-continuous Tietze extension)

Suppose $Y \subseteq X$ are G-flows and that $f \in \mathrm{C}_b(Y)$ is d-orbit-Lipschitz. Is there $f' \in \mathrm{C}_b(Y)$ extending f which is d'-orbit-Lipschitz, where d' only depends on d?

A suggestive theorem with this flavor comes from work in topometric spaces, i.e. (compact) topological spaces equipped with a lower-semicontinuous metric.

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Theorem (Ben Yaacov 2013)

If (X, ∂) is a compact topometric space, $Y \subseteq X$ is a closed subspace, and $f \in \mathrm{C}_b(Y)$ is ∂-Lipschitz, then for any $c > 1$, there is a $c\partial$-Lipschitz $f' \in \mathrm{C}_b(X)$ extending f.

Problem: Given a G-flow X and a continuous, right-invariant pseudometric d on G, the metric that d induces on X need not be lsc.

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A G-flow X is called weakly MHP if for every LRPC $d \in \mathrm{PsM}_r(G)$, the relation

$$\partial_d(x, y) \leq c \Leftrightarrow \forall A \ni_{op} x, B \ni_{op} y, \epsilon > 0 \, (\mathrm{B}_d(c + \epsilon) \cdot A \cap B \neq \emptyset)$$

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Every G-flow for G locally compact is weakly MHP. Every MHP G-flow is weakly MHP. The class of weakly MHP G-flows is closed under ultracoproducts.

Definition

Let $Y \subseteq X$ be G-flows. We call Y a fiber subflow of X if there is a motionless G-flow Z, a G-map $\pi\colon X \to Z$, and some $z_0 \in Z$ with $Y = \pi^{-1}(\{z_0\})$.

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Theorem (Z. 2023+)

If G is LRPC, X is a weakly MHP G-flow, and $Y \subseteq X$ is a weakly MHP fiber subflow, then for any $d \in \mathrm{PsM}_r(G)$, $c > 1$, and d-orbit-Lipschitz $f \in \mathrm{C}_b(Y)$, there is $f' \in \mathrm{C}_b(X)$ extending f which is cd-orbit-Lipschitz.

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The proof uses Ben Yaacov's topometric Tietze extension theorem as well as a relativized version of the characterization of RPC groups as those with $\mathrm{TT}_G(X)$ Vietoris closed.

Corollary (Z. 2023+)

If G is an LRPC group, then on the class of weakly MHP G-flows, weak containment is a pre-order and weak equivalence is an equivalence relation.

In particular, if G is locally compact, then this holds for the class of all G-flows.

Thanks!